ARCHER2-PROJECT-eCSE11-7

eCSE Title: ***"High Performance Algorithms for the Computation of the Hardy Function – Dissemination & Development"***

PI Name and Speaker: Dr D M Lewis (University of Liverpool).

Technical Staff: Dr Mario Antonioletti (EPCC, The University of Edinburgh),

Dr A. R. Brereton (University of California, Berkeley).

Webinar Title:

New, high-performance algorithms for the computation of the Hardy Function

Date: 23rd April 2025

Webinar Abstract

The Riemann zeta-function $\zeta(s)$ for $s \in \mathbb{C}$, is a complex function with a deep connection to the distribution of prime numbers and other areas of mathematics and physics. Its computation along the critical line $s = 1/2 + \mathrm{I}t,\ t \in \mathbb{R}$, is particularly relevant to number theory and the study of the distribution of primes, since this is where all the (non-trivial) zeta-zeros are predicted (by the *Riemann Hypothesis*) to occur. Hardy's Z function $Z(t) = e^{\mathrm{I}\theta(t)}\zeta(1/2 + \mathrm{I}t),\ \theta(t) \in \mathbb{R}$, defines the amplitude of zeta-function along the critical line. Computation of $Z(t)$ is important not only in the determination of zeta-zeros and *Riemann Hypothesis* verification within closed intervals, but also in the determination of bounds on the magnitude of $\zeta(s)$ along the critical line, central to the *Lindelöf Hypothesis.*

During the 20th century, the most efficient means of computing $Z(t)$ was by means of the classical Riemann-Siegel (RS) formula, an $O(\sqrt{t})$ operational method. Although still extremely important today, the RS formulation has been superseded as a means of computing $Z(t)$ for large t values by an algorithm devised by G. A. Hiary, which requires just $O\left(t^{1/3}\{log(t)\}^{\kappa}\right)$ operations. The methodology involved the sub-division of the RS formula into sequences of quadratic Gauss sums of various lengths N. Such sums are amenable to rapid computation, in $O(log(N))$ operations, using a standard recursive scheme. This webinar will present some significant advancement of these ideas. Central to these advances is a new asymptotic formula for $Z(t)$ with some interesting analytical properties. From a purely computational viewpoint, this new formulation allows one to go beyond the quadratic sum expansion and express $Z(t)$ in terms of more complex sub-sequences of generalised, m^{th}-order, Gauss sums, of progressively increasing length. At least for the cubic $m = 3$ case, these sums can also be computed rapidly, utilising a similar, but more powerful, recursive scheme to that implemented for the quadratic case. The net result is a computational algorithm for $Z(t)$ with more efficient, $O\left((t/\varepsilon_t)^{1/4}\{log(t)\}^3\right)$ operational count, to high accuracy. Sample computations, obtained from the open-source code developed as part of this project and now available on GitHub repository, support these findings.